Computation of Extrinsic Magneto-Electric Problem using E-H formulation

Thu Trang NGUYEN, Frédéric BOUILLAULT, Xavier MININGER and Laurent DANIEL
Laboratoire de Génie Electrique de Paris, CNRS UMR8507; SUPELEC; UPMC Univ Paris 06; Univ Paris-Sud
11 rue Joliot-Curie, Plateau de Moulon, Gif-sur-Yvette, France
thutrang.nguyen@supelec.fr

***Abstract*—This paper deals with the modeling of the magnetoelectric effect in composite material. This work focuses on the modeling of a magnetoelectric bilayer using a full 2D finite element model. Thanks to this model, the magnetic induction in the direction normal to the plan - neglected in other 2D formulations - can be investigated.**

I. Introduction

The magnetoelectric (ME) phenomenon consists in the existence of a magnetization induced by an electric polarization, or conversely an electric polarization induced by a magnetization. In this paper, we focus on the modeling of magneto-electric laminate composites using two-dimensional finite element method. The magneto-electric laminate composite consists in layered piezoelectric and magnetostrictive materials that produces the ME effect through an elastic interaction. Previous papers have described different formulations for magnetoelectric effect for two dimensional problems: The use of electric and magnetic scalar potentials has been proposed by Liu *et al.* [1]. Galopin *et al.* [2] take electric scalar and magnetic vector potentials as variables. In the harmonic case, the ME effect is modified because of the resonance of the mechanical structure [3] and the appearance of electromagnetic coupling based on Maxwell equations. In order to maintain an electric scalar potential formulation, the assumption of no magnetic induction in the direction normal to the working plane (*z*-direction) is made. This assumption has to be justified because of the presence of electric field in piezoelectric element creating a magnetic induction along *z*-direction.
The aim of this paper is to build a full 2D finite element formulation taking into account the electromagnetic coupling effect in order to discuss the classical model using magnetic and electric potentials.

II. Constitutive Laws

The working plane is defined on the *xy*-cordinates plane, *z*-direction is normal to the working plane. In the formulation, we denote by $\mathbf{T}$ the stress tensor, $\mathbf{S}$ the strain tensor, $\mathbf{u}$ the displacement, $\mathbf{E}$ the electric field, $\mathbf{D}$ the electric flux density, σ the electric conductivity, $\mathbf{H}$ the magnetic field, $\mathbf{B}$ the magnetic induction. We note $\widetilde{\mathrm{X}}(\widetilde{\mathrm{a}},\widetilde{\mathrm{b}})$ the small variation of X around a polarization point $\mathrm{X}_0(\mathrm{a}_0,\mathrm{b}_0)$:

$$\widetilde{X} = \frac{\partial X}{\partial a}(a_0,b_0)\widetilde{a} + \frac{\partial X}{\partial b}(a_0,b_0)\widetilde{b} \qquad X = X_0 + \widetilde{X} \tag{1}$$

A. Linearization of magnetostrictive and piezoelectric coefficients

1) Magnetostrictive coefficients: We assume that the magnetostriction phenomenon is isochoric and isotropic, and that the magnetostriction strain $\mathbf{s}^\mu$ can be expressed as a parabolic function of the magnetic induction.We can then write [2]:

$$s^\mu_{ij} = \frac{\beta}{2}(3b_ib_j - \delta_{ij}b_kb_k) \tag{2}$$

where δ_{ij} is the Kronecker symbol.
We consider that the polarization due to the applied static magnetic induction is along *x*-axis, thus the variation of magnetostrictive strain imposed by the variation of magnetic induction $\mathbf{B}$ ($\mathbf{B} = B_0\mathbf{x} + \widetilde{\mathbf{B}}$) can be calculated as:

$$\widetilde{\mathbf{s}}^\mu_{vec} = B_0\beta_0\begin{pmatrix}2\widetilde{b_x} & -\widetilde{b_x} & -\widetilde{b_x} & 0 & 3\widetilde{b_z} & 3\widetilde{b_y}\end{pmatrix}^t \tag{3}$$

where $\mathbf{s}_{vec}$ denotes the Voigt notation of the strain $\mathbf{s}$: $\mathbf{s}_{vec} = \begin{pmatrix}s_{11} & s_{22} & s_{33} & 2s_{23} & 2s_{31} & 2s_{12}\end{pmatrix}^t$

2) Piezoelectric coefficients: We consider that the polarization of the piezoelectric element is along *y*-axis, a similar expression of piezoelectric strain $\mathbf{s}^p$ imposed by the variation of electric displacement field $\mathbf{D}$ ($\mathbf{D} = D_0\mathbf{y} + \widetilde{\mathbf{D}}$) can be deduced:

$$\widetilde{\mathbf{s}}^p_{vec} = D_0\alpha_0\begin{pmatrix}-\widetilde{d_y} & 2\widetilde{d_y} & -\widetilde{d_y} & 3\widetilde{d_z} & 0 & 3\widetilde{d_x}\end{pmatrix}^t \tag{4}$$

B. Mechanical assumptions

Using Lamé coefficients μ^* and λ^* in the case of isotropic material, the total stress $\mathbf{t}$ is expressed by:

$$\mathbf{t} = 2\mu^*\mathbf{s}^e + \lambda^* tr(\mathbf{s}^e)\,\mathbf{I} \tag{5}$$

where $\mathbf{s}^e = \mathbf{s} - \mathbf{s}^c$ is the elastic strain, $\mathbf{s}^c = \mathbf{s}^\mu$ in the magnetostrictive material (MM), $\mathbf{s}^c = \mathbf{s}^p$ in the piezoelectric material (PM). $\mathbf{I}$ is the identity second order tensor. In this paper we consider plane stress conditions ($t_{31} = t_{32} = t_{33} = 0$) leading to the following relations:

$$\begin{cases} s^e_{31} = s^e_{32} = 0 \\ s^e_{33} = \dfrac{\lambda^*}{2\mu^* + \lambda^*}(s^e_{11} + s^e_{22}) \end{cases} \tag{6}$$

From equations (3), (4) and (6), the stress along *z*-direction can be calculated from the stress in the working plane:

$$s_{31} = \begin{cases} 0 \text{ in PM} \\ \dfrac{3}{2}B_0\beta_0 b_z \text{ in MM} \end{cases} \qquad s_{32} = \begin{cases} \dfrac{3}{2}D_0\alpha_0 d_z \text{ in PM} \\ 0 \text{ in MM} \end{cases}$$
$$s_{33} = \frac{\lambda^*}{2\mu^* + \lambda^*}(s_{11} + s_{22}) - \frac{2\mu^*}{2\mu^* + \lambda^*}G\widetilde{v} \tag{7}$$

where $G = D_0\alpha_0$, $\widetilde{v} = \widetilde{d_y}$ in PM, $G = B_0\beta_0$, $\widetilde{v} = \widetilde{b_x}$ in MM.

C. Constitutive laws

Considering $\mathbf{S}$, $\mathbf{D}$ and $\mathbf{B}$ as state variables, at a polarization point of PM or MM, using the thermodynamical approach [4], and calculating the differentials of Hooke's law [5], the linearized forms of the constitutive laws of PM and MM are defined in 3D as:

$$\begin{aligned}\widetilde{t}_{ij} &= C_{ijkl}\widetilde{s}_{kl} - \gamma_{ijk}\widetilde{v}_k \\ \widetilde{r}_i &= -\gamma_{ijk}\widetilde{s}_{jk} + \tau_{ij}\widetilde{v}_j\end{aligned} \qquad (8)$$

where C_{ijkl} is the stiffness tensor.
In MM: $\boldsymbol{\gamma} = \boldsymbol{\beta}$ the magnetostrictive coupling matrix, $\boldsymbol{\tau} = \boldsymbol{\nu}_{\text{eff}}$ with $\boldsymbol{\nu}_{\text{eff}}$ the effective reluctivity, $\widetilde{\mathbf{v}} = \widetilde{\mathbf{b}}$, $\widetilde{\mathbf{r}} = \widetilde{\mathbf{h}}$.
In PM: $\boldsymbol{\gamma} = \boldsymbol{\alpha}$ the piezoelectric coupling matrix, $\boldsymbol{\tau} = \boldsymbol{\epsilon}_{\text{eff}}^{-1}$ with $\boldsymbol{\epsilon}_{\text{eff}}$ the effective permittivity, $\widetilde{\mathbf{v}} = \widetilde{\mathbf{d}}$, $\widetilde{\mathbf{r}} = \widetilde{\mathbf{e}}$.
The constitutive laws presented in equation (8) can be simplified in the case of 2D. From equations (3), (4), the coupling matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ can be deduced from α_0, β_0, D_0 and B_0 depending on the polarization points. From the mechanical assumptions (equation (7)), mechanical unknows along z can be calculated afterwards from mechanical unknows in xy-plane. Consequently, from equation (8), $\mathbf{E}$ and $\mathbf{H}$ can be calculated separately in xy-plane ($/\!\!/$) and along z ($\perp$):

$$\begin{aligned}\widetilde{e}_\perp &= \epsilon_\perp^{-1}\widetilde{d}_\perp & \widetilde{h}_\perp &= \nu_\perp\widetilde{b}_\perp \\ \widetilde{\mathbf{E}}_{/\!/} &= \boldsymbol{\epsilon}_{/\!/}^{-1}\widetilde{\mathbf{D}}_{/\!/} - \boldsymbol{\alpha}_{/\!/}\widetilde{\mathbf{S}}_{/\!/} & \widetilde{\mathbf{H}}_{/\!/} &= \boldsymbol{\nu}_{/\!/}\widetilde{\mathbf{B}}_{/\!/} - \boldsymbol{\beta}_{/\!/}\widetilde{\mathbf{S}}_{/\!/} \\ \widetilde{\mathbf{T}}_{/\!/} &= \mathbf{C}_{/\!/}\widetilde{\mathbf{S}}_{/\!/} - \boldsymbol{\alpha}_{/\!/}\widetilde{\mathbf{D}}_{/\!/} & \widetilde{\mathbf{T}}_{/\!/} &= \mathbf{C}_{/\!/}\widetilde{\mathbf{S}}_{/\!/} - \boldsymbol{\beta}_{/\!/}\widetilde{\mathbf{B}}_{/\!/}\end{aligned} \qquad (9)$$

where $\epsilon_\perp$ and $\nu_\perp$ are the equivalent permittivity and reluctivity along z-axis.

III. Finite Element Formulation

A. Electromagnetic equations

3D magnetoelectric coupling problems are governed by Maxwell equations:

$$\mathbf{rotE} = -\partial_t\mathbf{B} \quad \mathbf{rotH} = \sigma\mathbf{E} + \partial_t\mathbf{D} \qquad (10)$$

In a 2D harmonic problem, $\mathbf{E}$ and $\mathbf{H}$ do not depend on z, thus each equation (10) can be divided into 2 equations:

$$\begin{aligned}\mathbf{r}^*\mathbf{grad}e_\perp &= -j\omega\mathbf{B}_{/\!/} & \mathbf{r}^*\mathbf{grad}h_\perp &= \sigma\mathbf{E}_{/\!/} + j\omega\mathbf{D}_{/\!/} \\ \text{div}(\mathbf{r}^*\mathbf{E}_{/\!/}) &= -j\omega b_\perp & \text{div}(\mathbf{r}^*\mathbf{H}_{/\!/}) &= \sigma e_\perp + j\omega d_\perp\end{aligned} \qquad (11)$$

where $\mathbf{r}^*$ is defined as $\mathbf{r}^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
Noting $a^* = -\dfrac{e_\perp}{j\omega}$ and $t^* = \dfrac{h_\perp}{j\omega}$, $\mathbf{I}$ the indentity matrix, using the constitutive laws (9), the system to be solved is:

$$\begin{aligned}&\text{div}(\mathbf{r}^*(\boldsymbol{\epsilon}_{/\!/} + \frac{\sigma}{j\omega}\mathbf{I})^{-1}\mathbf{r}^*\mathbf{grad}t^* - \mathbf{r}^*\boldsymbol{\alpha}_{/\!/}^t\mathbf{S}_{/\!/}) = \omega^2\boldsymbol{\nu}_\perp^{-1}t^* \\ &\text{div}(\mathbf{r}^*\boldsymbol{\nu}_{/\!/}\mathbf{r}^*\mathbf{grad}a^* - \mathbf{r}^*\boldsymbol{\beta}_{/\!/}^t\mathbf{S}_{/\!/}) = (-j\omega\sigma + \omega^2\epsilon_\perp)a^*\end{aligned} \qquad (12)$$

B. Mechanical equation

The mechanical equilibrium reads: $\text{div}\mathbf{T} = -\mathbf{f}$. From equation (9) and (11) the mechanical equation to be solved in 2D problem is:

$$\begin{aligned}&\text{div}(\mathbf{C}_{/\!/}\mathbf{S}_{/\!/} - \boldsymbol{\alpha}_{/\!/}\mathbf{r}^*\mathbf{grad}t^*) = -\mathbf{f} \quad \text{(PM)} \\ &\text{div}(\mathbf{C}_{/\!/}\mathbf{S}_{/\!/} - \boldsymbol{\beta}_{/\!/}\mathbf{r}^*\mathbf{grad}a^*) = -\mathbf{f} \quad \text{(MM)}\end{aligned} \qquad (13)$$

As $\mathbf{S}_{/\!/} = \dfrac{1}{2}(\mathbf{gradu}_{/\!/} + {}^t\mathbf{gradu}_{/\!/})$, the variables of the system to be solved are $\mathbf{u}_{/\!/}$, t^*, a^*. Compared to the formulation using $\mathbf{u}_{/\!/}$, the electric and magnetic potentials (V, a or T), the new formulation takes the same computation cost: 4 unknowns per node. Moreover, to deal with the electrodes on PM, we need a further processing to be detailed in the full paper.

IV. Numerical Example & Conclusion

In this section, we study a ME bilayer (Figure 1) pre-polarized by a static magnetic field $\mathbf{H}_{\text{dc}}$. An harmonic magnetic field $\mathbf{h}_{\text{ac}}$ is applied in order to obtain an electric voltage v_{ac}. This configuration corresponds to a magnetic sensor [6].

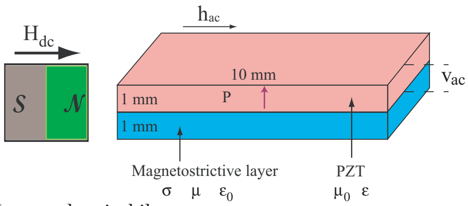

Fig. 1. Magnetoelectric bilayer

As the MM is an electrical conductor, the working frequency is 100 kHz in order to observe the skin effect. The skin depth theoretically calculated is about 0.25 mm.

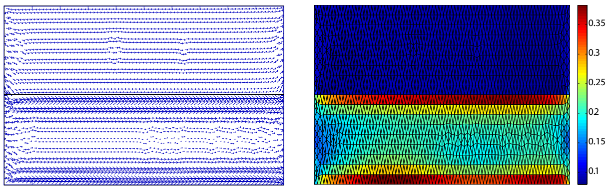

Fig. 2. Magnetic induction $\mathbf{B}_{/\!/}$ in xy-plane: Vector and Value

We observe also the electric field in PM related to the magnetic induction along z-direction according to Faraday's law (Figure 3).

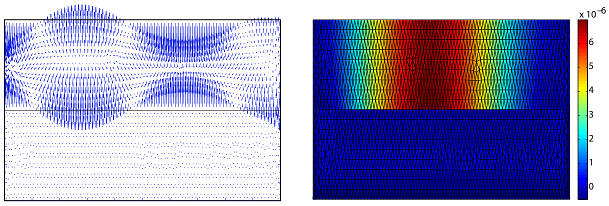

Fig. 3. Electric field $\mathbf{E}_{/\!/}$ in xy-plane vs Magnetic induction $\mathbf{B}_z$ in z

The result in Figure 3 confirms the existence of a magnetic induction along z-axis in PM. Nevertheless, its value in that case is negligible compared to the magnetic induction in xy-plane. Considering no magnetic induction along z-axis is therefore a justified assumption. The new formulation justifies the use of the classical formulation. It needs the same computation cost and gives more accurate results. This new formulation will be used for the comparison with a 3D-formulation in progress.

References

[1] Y. X. Liu *et al*, J. Appl. Phys, 94, 5111-5117, 8(2003).
[2] N. Galopin *et al*, IEEE Trans. Magn., vol.44, no.6, pp.834-837, 6(2008).
[3] M. I. Bichurin *et al*, Phys. Rev. B, vol.68, 132408, 10(2003).
[4] M. Besbes *et al*, IEEE Trans. Magn., vol. 37, no. 1, pp. 3324-3328, 2001.
[5] T. T. Nguyen *et al*, IEEE Trans. Magn., CEFC 2010 (to be published).
[6] G. T. Do *et al*, Sensor and Actuator A, vol. 149, 229–232, 2009.